

LEGENDRE DUALITY BETWEEN LAGRANGIAN AND HAMILTONIAN MECHANICS

by
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Abstract

In some previous papers, a Legendre duality between Lagrangian and Hamiltonian Mechanics has been developed. The (ρ, η) -tangent application of the Legendre bundle morphism associated to a Lagrangian L or Hamiltonian H is presented. Using that, a Legendre description of Lagrangian Mechanics and Hamiltonian Mechanics is developed. Duality between Lie algebroids structure, adapted (ρ, η) -basis, distinguished linear (ρ, η) -connections and mechanical (ρ, η) -systems is the scope of this paper. In the particular case of Lie algebroids, new results are presented. In the particular case of the usual Lie algebroid tangent bundle, the classical results are obtained.

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1 Introduction

The notion of Lagrange space was introduced and studied by J. Kern [14] and R. Miron [20]. In the Classical Mechanics we obtain a Lagrangian formalism if we use TTM as space for developing the theory. (see [6, 7, 21, 22, 26, 27, 28, 34])

In [18], P. Liberman showed that such a Lagrangian formalism is not possible if the tangent bundle of a Lie algebroid is considered as space for developing the theory. In his paper [39] A. Weinstein developed a generalized theory of Lagrangian Mechanics on Lie algebroids and obtained the equations of motion, using the linear Poisson structure on the dual of the Lie algebroid and the Legendre transformation associated with the regular Lagrangian L . In that paper, he asks the question of whether it is possible to develop a formalism on Lie algebroids similar to Klein's formalism [16] in ordinary Lagrangian Mechanics. This task was finally done by E. Martinez in [19] (see also [9, 18]).

The geometry of algebroids was extensively studied by many authors. (see [3, 29, 30,], [35, 36, 37, 38])

Using the generalized Lie algebroids, (see [1, 2]) a new point of view over the Lagrangian Mechanics is presented in the paper [4]. The canonical (ρ, η) -(semi)spray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta) \Gamma)$ and from locally invertible \mathbf{B}^V -morphism (g, h) have been presented. Lagrange mechanical (ρ, η) -systems, the spaces necessary to solve the Weinstein's problem, were introduced. There have been presented the (ρ, η) -semispray associated to a regular Lagrangian L and external force F_e which are applied on the total space of a generalized Lie algebroid and the equations of Euler-Lagrange type.

The geometry of T^*M is from one point of view different from that of TM , because it does not exist a natural tangent structure and a semispray can not be introduced in the case of the tangent bundle. Two geometrical ingredients are of great importance to T^*M : the canonical 1-form $p_i dx^i$ and its exterior derivative $dp_i \wedge dx^i$ (the canonical symplectic structure of T^*M). They are systematically used to define new useful tools in the classical theory.

The concept of Hamilton space, introduced in [25] and intensively studied in [10, 11,] [12, 13, 21, 22] has been succesful, as a geometric theory of the Hamiltonian function. The modern formulation of the geometry of Cartan spaces was given by R. Miron [23, 24] although some results were obtained by É. Cartan [8] and A. Kawaguchi [15].

A Hamiltonian description of Mechanics on duals of Lie algebroids was presented in [17]. (see also [32, 33, 35, 36, 37, 38]) The role of cotangent bundle of the configuration manifold is played by the prolongation $\mathcal{L}^{\tau^*} E$ of E along the projection $E^* \xrightarrow{\tau^*} M$. The Lie algebroid version of the classical results concerning the universality of the standard Liouville 1-form on cotangent bundles is presented. Given a Hamiltonian function $E^* \xrightarrow{H} \mathbb{R}$ and the symplectic form Ω_E on E^* , the dynamics are obtained solving the equation

$$i_{\xi_H} \Omega_E = d\mathcal{L}^{\tau^*} E H$$

with the usual notations. The solutions of ξ_H (curves in E^*) are the ones of the Hamilton equations for H . (see [17])

A Hamiltonian description of Mechanics on duals of generalized Lie algebroids without the symplectic form is presented in the paper [5]. The canonical (ρ, η) -semispray associated to the dual mechanical (ρ, η) -system $\left(\left(E^*, \pi^*, M \right), F_e^*, (\rho, \eta) \Gamma \right)$ and from lo-

cally invertible \mathbf{B}^\vee -morphism (g, h) . Also, the canonical (ρ, η) -spray associated to mechanical system $\left(\left(E, \pi, M\right), \overset{*}{F}_e, (\rho, \eta) \Gamma\right)$ and from locally invertible \mathbf{B}^\vee -morphism (g, h) have been presented.

The Hamilton mechanical (ρ, η) -systems are the spaces necessary to obtain a Hamiltonian formalism in the general framework of generalized Lie algebroids. The (ρ, η) -semispray associated to a regular Hamiltonian H and external force $\overset{*}{F}_e$ which are applied on the dual of the total space of a generalized Lie algebroid and the equations of Hamilton-Jacobi type have been presented.

Using the classical Legendre transformation different geometrical objects on TM are naturally related to similar ones on T^*M . The geometry of Hamilton space can be obtained from that of a certain Lagrange space and vice versa. As a particular case, we can associate its dual to a given Finsler which is a Cartan space. In addition, in some conditions the L -dual of Kropina space is the Randers space and the L -dual of Randers space is the Kropina space. These spaces are used in several applications in Physics. (see [13])

The Legendre transformation $E \xrightarrow{Leg_L} E^*$ associated with a Lagrangian L induces a Lie algebroid morphism $\mathcal{L}^\tau E \xrightarrow{\mathcal{L}Leg_L} \mathcal{L}^{\tau^*} E$, which permits in the regular case to connect Lagrangian and Hamiltonian formalisms as in Classical Mechanics. (see [17])

In this paper we extend our study in the general framework of generalized Lie algebroids. Using the (ρ, η) -tangent application of the Legendre bundle morphism associated to a Lagrangian L or Hamiltonian H we obtain a lot of new results.

The Lagrangian Mechanics presented in the paper [4] is dual to the Hamiltonian Mechanics presented in the paper [5] and vice versa. In particular, a new point of view over the Legendre duality between Lagrangian Mechanics and Hamiltonian Mechanics in the framework of Lie algebroids is presented in this paper.

2 Preliminaries

Let **Vect**, **Liealg**, **Mod**, **Man**, **B** and \mathbf{B}^\vee be the category of real vector spaces, Lie algebras, modules, manifolds, fiber bundles and vector bundles respectively.

We know that if $(E, \pi, M) \in |\mathbf{B}^\vee|$ so that M is paracompact and if $A \subseteq M$ is closed, then for any section u over A it exists $\tilde{u} \in \Gamma(E, \pi, M)$ so that $\tilde{u}|_A = u$. In the following, we consider only vector bundles with paracompact base.

Additionally, if $(E, \pi, M) \in |\mathbf{B}^\vee|$, $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$ and $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$, then $(\Gamma(E, \pi, M), +, \cdot)$ is a $\mathcal{F}(M)$ -module. If $(\varphi, \varphi_0) \in \mathbf{B}^\vee((E, \pi, M), (E', \pi', M'))$ such that $\varphi_0 \in Iso_{\mathbf{Man}}(M, M')$, then, using the operation

$$\begin{aligned} \mathcal{F}(M) \times \Gamma(E', \pi', M') &\longrightarrow \Gamma(E', \pi', M') \\ (f, u') &\longmapsto f \circ \varphi_0^{-1} \cdot u' \end{aligned}$$

it results that $(\Gamma(E', \pi', M'), +, \cdot)$ is a $\mathcal{F}(M)$ -module and we obtain the **Mod**-morphism

$$\begin{aligned} \Gamma(E, \pi, M) &\xrightarrow{\Gamma(\varphi, \varphi_0)} \Gamma(E', \pi', M') \\ u &\longmapsto \Gamma(\varphi, \varphi_0) u \end{aligned}$$

defined by

$$\Gamma(\varphi, \varphi_0) u(y) = \varphi(u_{\varphi_0^{-1}(y)}),$$

for any $y \in M'$.

Let $M, N \in |\mathbf{Man}|$, $h \in Iso_{\mathbf{Man}}(M, N)$ and $\eta \in Iso_{\mathbf{Man}}(N, M)$ be.

We know (see [1, 2]) that if $(F, \nu, N) \in |\mathbf{B}^{\mathbf{v}}|$ so that there exists

$$(\rho, \eta) \in \mathbf{B}^{\mathbf{v}}((F, \nu, N), (TM, \tau_M, M))$$

and also an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_{F,h}} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_{F,h} \end{array}$$

with the following properties:

GLA_1 . the equality holds good

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v,$$

for all $u, v \in \Gamma(F, \nu, N)$ and $f \in \mathcal{F}(N)$.

GLA_2 . the 4-tuple $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h})$ is a Lie $\mathcal{F}(N)$ -algebra,

GLA_3 . the **Mod**-morphism $\Gamma(Th \circ \rho, h \circ \eta)$ is a **LieAlg**-morphism of

$$(\Gamma(F, \nu, N), +, \cdot, [\cdot]_{F,h})$$

source and

$$(\Gamma(TN, \tau_N, N), +, \cdot, [\cdot]_{TN})$$

target, then the triple $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ is called generalized Lie algebroid.

In particular, if $h = Id_M = \eta$, then we obtain the definition of the Lie algebroid.

Let $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$ be a generalized Lie algebroid.

- Locally, for any $\alpha, \beta \in \overline{1, p}$, we set $[t_\alpha, t_\beta]_{F,h} = L_{\alpha\beta}^\gamma t_\gamma$. We easily obtain that $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$, for any $\alpha, \beta, \gamma \in \overline{1, p}$.

The real local functions $L_{\alpha\beta}^\gamma$, $\alpha, \beta, \gamma \in \overline{1, p}$ will be called the *structure functions of the generalized Lie algebroid* $((F, \nu, N), [\cdot]_{F,h}, (\rho, \eta))$.

- We assume the following diagrams:

$$\begin{array}{ccccc} F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\ \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \end{array}$$

$$(\chi^{\tilde{i}}, z^\alpha) \qquad (x^i, y^i) \qquad (\chi^{\tilde{i}}, z^{\tilde{i}})$$

where $i, \tilde{i} \in \overline{1, m}$ and $\alpha \in \overline{1, p}$.

If

$$(\chi^{\tilde{i}}, z^\alpha) \longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\alpha'}(\chi^{\tilde{i}}, z^\alpha)),$$

$$(x^i, y^i) \longrightarrow (x^{\check{i}}(x^i), y^{\check{i}}(x^i, y^i))$$

and

$$(\chi^{\tilde{i}}, z^{\tilde{i}}) \longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\tilde{i}'}(\chi^{\tilde{i}}, z^{\tilde{i}})),$$

then

$$z^{\alpha'} = \Lambda_{\alpha}^{\alpha'} z^{\alpha},$$

$$y^{\check{i}} = \frac{\partial x^{\check{i}}}{\partial x^i} y^i$$

and

$$z^{\tilde{i}'} = \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}} z^{\tilde{i}}.$$

- We assume that $(\theta, \mu) \stackrel{put}{=} (Th \circ \rho, h \circ \eta)$. If $z^{\alpha} t_{\alpha} \in \Gamma(F, \nu, N)$ is arbitrary, then

$$(2.1) \quad \begin{aligned} & \Gamma(Th \circ \rho, h \circ \eta)(z^{\alpha} t_{\alpha}) f(h \circ \eta(\varkappa)) = \\ & = \left(\theta_{\alpha}^{\tilde{i}} z^{\alpha} \frac{\partial f}{\partial \varkappa^{\tilde{i}}} \right) (h \circ \eta(\varkappa)) = \left((\rho_{\alpha}^i \circ h)(z^{\alpha} \circ h) \frac{\partial f \circ h}{\partial x^i} \right) (\eta(\varkappa)), \end{aligned}$$

for any $f \in \mathcal{F}(N)$ and $\varkappa \in N$.

The coefficients ρ_{α}^i respectively $\theta_{\alpha}^{\tilde{i}}$ change to $\rho_{\alpha'}^{\check{i}}$ respectively $\theta_{\alpha'}^{\tilde{i}'}$ according to the rule:

$$(2.2) \quad \rho_{\alpha'}^{\check{i}} = \Lambda_{\alpha}^{\alpha'} \rho_{\alpha}^i \frac{\partial x^{\check{i}}}{\partial x^i},$$

respectively

$$(2.3) \quad \theta_{\alpha'}^{\tilde{i}'} = \Lambda_{\alpha}^{\alpha'} \theta_{\alpha}^{\tilde{i}} \frac{\partial \varkappa^{\tilde{i}'}}{\partial \varkappa^{\tilde{i}}},$$

where

$$\|\Lambda_{\alpha'}^{\alpha}\| = \|\Lambda_{\alpha}^{\alpha'}\|^{-1}.$$

Remark 2.1 The following equalities hold good:

$$(2.4) \quad \rho_{\alpha}^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left(\theta_{\alpha}^{\tilde{i}} \frac{\partial f}{\partial \varkappa^{\tilde{i}}} \right) \circ h, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad (L_{\alpha\beta}^{\gamma} \circ h) (\rho_{\gamma}^k \circ h) = (\rho_{\alpha}^i \circ h) \frac{\partial (\rho_{\beta}^k \circ h)}{\partial x^i} - (\rho_{\beta}^j \circ h) \frac{\partial (\rho_{\alpha}^k \circ h)}{\partial x^j}.$$

3 Legendre transformation

Let (E, π, M) be a vector bundle. We take (x^i, y^a) as canonical local coordinates on (E, π, M) , where $i \in \overline{1, m}$ and $a \in \overline{1, r}$. Consider

$$(x^i, y^a) \longrightarrow (x^{\check{i}}(x^i), y^{\check{a}}(x^i, y^a))$$

a change of coordinates on (E, π, M) . Then the coordinates y^a change to $y^{a'}$ according to the rule:

$$(3.1) \quad y^{a'} = M_a^{a'} y^a.$$

Let $\left(\partial_i, \dot{\partial}_a\right)$ be the natural base of the Lie $\mathcal{F}(E)$ -algebra $(\Gamma(T E, \tau_E, E), +, \cdot, [,]_{T E})$.

Let $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ be the dual vector bundle of (E, π, M) . We take (x^i, p_a) as canonical local coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$, where $i \in \overline{1, m}$ and $a \in \overline{1, r}$. Consider

$$(x^i, p_a) \longrightarrow (x^{\check{i}}(x^i), p_{a'}(x^i, p_a))$$

a change of coordinates on $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$. Then the coordinates p_a change to $p_{a'}$ according to the rule:

$$(3.2) \quad p_{a'} = M_a^{a'} p_a.$$

If (U, s_U) and $\left(U, \overset{*}{s}_U\right)$ are vector local $(m+r)$ -charts then

$$M_a^a(x) \cdot M_b^{a'}(x) = \delta_b^a, \quad \forall x \in U.$$

Let $\left(\overset{*}{\partial}_i, \overset{*}{\partial}^a\right)$ be the natural base of the Lie $\mathcal{F}\left(\overset{*}{E}\right)$ -algebra $\left(\Gamma\left(\overset{*}{T E}, \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot, [,]_{\overset{*}{T E}}\right)$.

Let L be a Lagrangian on the total space of the vector bundle (E, π, M) . (see [4]) If (U, s_U) is a vector local $(m+r)$ -chart for (E, π, M) , then we obtain the following real functions defined on $\pi^{-1}(U)$:

$$(3.3) \quad \begin{aligned} L_i &\stackrel{put}{=} \frac{\partial L}{\partial x^i} & L_{ib} &\stackrel{put}{=} \frac{\partial^2 L}{\partial x^i \partial y^b} \\ L_a &\stackrel{put}{=} \frac{\partial L}{\partial y^a} & L_{ab} &\stackrel{put}{=} \frac{\partial^2 L}{\partial y^a \partial y^b} \end{aligned}.$$

We build the fiber bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{\varphi_L} & \overset{*}{E} \\ \pi \downarrow & & \downarrow \overset{*}{\pi} \\ M & \xrightarrow{Id_M} & M \end{array}$$

where φ_L is locally defined

$$(3.4) \quad \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_L} & \overset{*}{\pi}^{-1}(U) \\ u_x & \longmapsto & L_a(u_x) s^a(x) \end{array},$$

for any vector local $(m+r)$ -chart (U, s_U) of (E, π, M) and for any vector local $(m+r)$ -chart $\left(U, \overset{*}{s}_U\right)$ of $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$.

We obtain the Hamiltonian H , locally defined by

$$(3.5) \quad \begin{array}{ccc} \overset{*}{\pi}^{-1}(U) & \xrightarrow{H} & \mathbb{R} \\ \overset{*}{u}_x = p_a s^a & \longmapsto & p_a y^a - L(u_x) \end{array},$$

for any vector local $(m+r)$ -chart (U, s_U) of (E, π, M) , where $(y^a, a \in \overline{1, r})$ are the components of the solutions of the differentiable equations

$$p_b = L_b(u_x), \quad u_x \in \pi^{-1}(U).$$

The Hamiltonian given by (3.5) will be called the *Legendre transformation of the Lagrangian* L .

If (U, s_U) is a vector local $(m+r)$ -chart for (E, π, M) , then we obtain the following real functions defined on $\pi^{-1}(U)$:

$$(3.3)' \quad \begin{aligned} H_i &= \frac{\partial H}{\partial x^i} & H_i^b &= \frac{\partial^2 H}{\partial x^i \partial p_b} \\ H^a &= \frac{\partial H}{\partial p_a} & H^{ab} &= \frac{\partial^2 H}{\partial p_a \partial p_b} \end{aligned}.$$

Using this Hamiltonian, we build the fiber bundle morphism

$$\begin{array}{ccc} \overset{*}{E} & \xrightarrow{\varphi_H} & E \\ \overset{*}{\pi} \downarrow & & \downarrow \pi \\ M & \xrightarrow{Id_M} & M \end{array},$$

where φ_H is locally defined

$$(3.4)' \quad \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_H} & \pi^{-1}(U) \\ \overset{*}{u}_x & \longmapsto & H^a(\overset{*}{u}_x) s_a(x) \end{array},$$

for any vector local $(m+r)$ -chart (U, s_U) of (E, π, M) and for any vector local $(m+r)$ -chart (U, s_U) of (E, π, M) .

We obtain that the Lagrangian L , is locally defined by

$$(3.5)' \quad \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{L} & \mathbb{R} \\ u_x = y^a s_a & \longmapsto & y^a p_a - H(\overset{*}{u}_x) \end{array},$$

for any vector local $(m+r)$ -chart (U, s_U) of (E, π, M) , where $(p_a, a \in \overline{1, r})$ are the components of the solutions of the differentiable equations

$$y^a = H^a(\overset{*}{u}_x), \quad \overset{*}{u}_x \in \pi^{-1}(U).$$

We will say that L is the *Legendre transformation of the Hamiltonian* H .

Remark 3.1 For any vector local $(m+r)$ -chart (U, s_U) of (E, π, M) and for any vector local $(m+r)$ -chart (U, s_U) of (E, π, M) we obtain:

$$(3.6) \quad \varphi_H \circ \varphi_L = Id_{\pi^{-1}(U)}$$

and

$$(3.7) \quad \varphi_L \circ \varphi_H = Id_{\pi^{-1}(U)}.$$

Therefore, locally, φ_L is diffeomorphism and $\varphi_L^{-1} = \varphi_H$.

4 Duality between Lie algebroids structures

Using the diagram:

$$(4.1) \quad \begin{array}{ccc} E & & (F, [,]_{F,h}, (\rho, \eta)) \\ \pi \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array},$$

where $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid, we build the Lie algebroid generalized tangent bundle (see [1, 4])

$$(4.2) \quad \left(((\rho, \eta) TE, (\rho, \eta) \tau_E, E), [,]_{(\rho, \eta) TE}, (\tilde{\rho}, Id_E) \right).$$

The natural (ρ, η) -base of sections is denoted

$$(4.3) \quad \left(\tilde{\partial}_\alpha, \dot{\tilde{\partial}}_a \right).$$

The Lie bracket $[\cdot, \cdot]_{(\rho, \eta) TE}$ is defined by

$$(4.4) \quad \begin{aligned} & \left[\left(Z_1^\alpha \tilde{\partial}_\alpha + Y_1^a \dot{\tilde{\partial}}_a \right), \left(Z_2^\beta \tilde{\partial}_\beta + Y_2^b \dot{\tilde{\partial}}_b \right) \right]_{(\rho, \eta) TE} = \\ & = \left[Z_1^\alpha T_a, Z_2^\beta T_\beta \right]_{\pi^*(h^*F)} \oplus \left[\left(\rho_\alpha^i \circ h \circ \pi \right) Z_1^\alpha \partial_i + Y_1^a \dot{\partial}_a, \right. \\ & \quad \left. \left(\rho_\beta^j \circ h \circ \pi \right) Z_2^\beta \partial_j + Y_2^b \dot{\partial}_b \right]_{TE}, \end{aligned}$$

for any sections $\left(Z_1^\alpha \tilde{\partial}_\alpha + Y_1^a \dot{\tilde{\partial}}_a \right)$ and $\left(Z_2^\beta \tilde{\partial}_\beta + Y_2^b \dot{\tilde{\partial}}_b \right)$.

The anchor map $(\tilde{\rho}, Id_E)$ is a \mathbf{B}^V -morphism of $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ source and (TE, τ_E, E) target, where

$$(4.5) \quad \begin{array}{ccc} (\rho, \eta) TE & \xrightarrow{\tilde{\rho}} & TE \\ \left(Z^\alpha \tilde{\partial}_\alpha + Y^a \dot{\tilde{\partial}}_a \right)(u_x) & \mapsto & \left(Z^\alpha (\rho_\alpha^i \circ h \circ \pi) \partial_i + Y^a \dot{\partial}_a \right)(u_x) \end{array}$$

Using the diagram:

$$(4.1)' \quad \begin{array}{ccc} E^* & & (F, [,]_{F,h}, (\rho, \eta)) \\ \pi^* \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array},$$

where $((F, \nu, N), [,]_{F,h}, (\rho, \eta))$ is a generalized Lie algebroid, we build the Lie algebroid generalized tangent bundle (see [1, 5])

$$(4.2)' \quad \left(\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right), [,]_{(\rho, \eta) TE^*}, \left(\tilde{\rho}^*, Id_{E^*} \right) \right).$$

The natural (ρ, η) -base of sections is denoted

$$(4.3)' \quad \left(\overset{*}{\tilde{\partial}}_\alpha, \overset{\cdot}{\tilde{\partial}}^a \right).$$

The Lie bracket $[\cdot, \cdot]_{(\rho, \eta)TE}^*$ is defined by

$$(4.4)' \quad \begin{aligned} & \left[\left(Z_1^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a^1 \overset{\cdot}{\tilde{\partial}}^a \right), \left(Z_2^\beta \overset{*}{\tilde{\partial}}_\beta + Y_b^2 \overset{\cdot}{\tilde{\partial}}^b \right) \right]_{(\rho, \eta)TE}^* = \\ & = \left[Z_1^\alpha T_a, Z_2^\beta T_\beta \right]_{\pi^* (h^* F)}^* \oplus \left[\left(\rho_\alpha^i \circ h \circ \pi^* \right) Z_1^\alpha \overset{*}{\partial}_i + Y_a^1 \dot{\partial}^a, \right. \\ & \quad \left. \left(\rho_\beta^j \circ h \circ \pi^* \right) Z_2^\beta \overset{*}{\partial}_j + Y_b^2 \dot{\partial}^b \right]_{TE}^*, \end{aligned}$$

for any sections $\left(Z_1^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a^1 \overset{\cdot}{\tilde{\partial}}^a \right)$ and $\left(Z_2^\beta \overset{*}{\tilde{\partial}}_\beta + Y_b^2 \overset{\cdot}{\tilde{\partial}}^b \right)$.

The anchor map $\left(\tilde{\rho}, Id_E^* \right)$ is a \mathbf{B}^v -morphism of $\left((\rho, \eta) TE, (\rho, \eta) \tau_E^*, E \right)$ source and $\left(TE, \tau_E^*, E \right)$ target, where

$$(4.5)' \quad \begin{array}{ccc} & (\rho, \eta) TE & \xrightarrow{\tilde{\rho}^*} TE \\ \left(Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot}{\tilde{\partial}}^a \right) (u_x^*) & \longmapsto & \left(Z^\alpha \left(\rho_\alpha^i \circ h \circ \pi^* \right) \overset{*}{\partial}_i + Y_a \dot{\partial}^a \right) (u_x^*) \end{array}$$

Using the \mathbf{B} -morphism (φ_L, Id_M) , we build the \mathbf{B}^v -morphism $((\rho, \eta) T\varphi_L, \varphi_L)$ given by the diagram

$$(4.6) \quad \begin{array}{ccc} (\rho, \eta) TE & \xrightarrow{(\rho, \eta) T\varphi_L} & (\rho, \eta) TE^* \\ (\rho, \eta) \tau_E \downarrow & & \downarrow (\rho, \eta) \tau_E^* \\ E & \xrightarrow{\varphi_L} & E^* \end{array}$$

such that

$$(4.7) \quad \begin{aligned} \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left(Z^\alpha \overset{*}{\tilde{\partial}}_\alpha \right) &= (Z^\alpha \circ \varphi_H) \overset{*}{\tilde{\partial}}_\alpha + [(\rho_\alpha^i \circ h \circ \pi) Z^\alpha L_{ib}] \circ \varphi_H \overset{\cdot}{\tilde{\partial}}^b, \\ \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left(Y^a \overset{\cdot}{\tilde{\partial}}_a \right) &= (Y^a L_{ab}) \circ \varphi_H \overset{\cdot}{\tilde{\partial}}^b, \end{aligned}$$

for any $Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y^a \overset{\cdot}{\tilde{\partial}}_a \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$, where H is the Legendre transformation of the Lagrangian L .

The \mathbf{B}^v -morphism $((\rho, \eta) T\varphi_L, \varphi_L)$ will be called the (ρ, η) -tangent application of the Legendre bundle morphism associated to the Lagrangian L .

Using the \mathbf{B} -morphism (φ_H, Id_M) , we build the \mathbf{B}^v -morphism $((\rho, \eta) T\varphi_H, \varphi_H)$ given by the diagram

$$(4.6)' \quad \begin{array}{ccc} (\rho, \eta) T\overset{*}{E} & \xrightarrow{(\rho, \eta) T\varphi_H} & (\rho, \eta) TE \\ (\rho, \eta) \tau_{\overset{*}{E}}^* \downarrow & & \downarrow (\rho, \eta) \tau_E \\ \overset{*}{E} & \xrightarrow{\varphi_H} & E, \end{array}$$

such that

$$(4.7)' \quad \begin{aligned} \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left(Z^\alpha \overset{*}{\tilde{\partial}}_\alpha \right) &= (Z^\alpha \circ \varphi_L) \tilde{\partial}_\alpha + \left[\left(\rho_\alpha^i \circ h \circ \pi^* \right) Z^\alpha H_i^b \right] \circ \varphi_L \overset{\cdot}{\tilde{\partial}}_b, \\ \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left(Y_a \overset{\cdot}{\tilde{\partial}}^a \right) &= (Y_a H^{ab}) \circ \varphi_L \overset{\cdot}{\tilde{\partial}}_b, \end{aligned}$$

for any $Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot}{\tilde{\partial}}^a \in \Gamma \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}^*, \overset{*}{E} \right)$.

The \mathbf{B}^v -morphism $((\rho, \eta) T\varphi_H, \varphi_H)$ will be called the (ρ, η) -tangent application of the Legendre bundle morphism associated to the Hamiltonian H .

Theorem 4.1 *If the \mathbf{B}^v -morphism $((\rho, \eta) T\varphi_L, \varphi_L)$ is morphism of Lie algebroids, then we obtain:*

$$(4.8) \quad \left(L_{\alpha\beta}^\gamma \circ h \circ \pi \right) \circ \varphi_H = L_{\alpha\beta}^\gamma \circ h \circ \pi^*,$$

$$(4.9) \quad \begin{aligned} \left[(L_{\alpha\beta}^\gamma \rho_\gamma^k) \circ h \circ \pi \cdot L_{kb} \right] \circ \varphi_H &= \rho_\alpha^i \circ h \circ \pi^* \cdot \frac{\partial}{\partial x^i} \left[(\rho_\beta^j \circ h \circ \pi \cdot L_{jb}) \circ \varphi_H \right] \\ &\quad - \rho_\beta^j \circ h \circ \pi^* \cdot \frac{\partial}{\partial x^j} \left[(\rho_\alpha^i \circ h \circ \pi \cdot L_{ib}) \circ \varphi_H \right] \\ &\quad + (\rho_\alpha^i \circ h \circ \pi \cdot L_{ia}) \circ \varphi_H \cdot \frac{\partial}{\partial p_a} \left[(\rho_\beta^j \circ h \circ \pi \cdot L_{jb}) \circ \varphi_H \right] \\ &\quad - (\rho_\beta^j \circ h \circ \pi \cdot L_{ja}) \circ \varphi_H \cdot \frac{\partial}{\partial p_a} \left[(\rho_\alpha^i \circ h \circ \pi \cdot L_{ib}) \circ \varphi_H \right], \end{aligned}$$

$$(4.10) \quad \begin{aligned} 0 &= \rho_\alpha^i \circ h \circ \pi^* \cdot \frac{\partial}{\partial x^i} (L_{ba} \circ \varphi_H) \\ &\quad + (\rho_\alpha^i \circ h \circ \pi \cdot L_{bc}) \circ \varphi_H \cdot \frac{\partial}{\partial p_c} (L_{ba} \circ \varphi_H) \\ &\quad - L_{bc} \circ \varphi_H \cdot \frac{\partial}{\partial p_c} \left[(\rho_\alpha^i \circ h \circ \pi \cdot L_{ia}) \circ \varphi_H \right] \end{aligned}$$

and

$$(4.11) \quad \begin{aligned} 0 &= L_{ac} \circ \varphi_H \cdot \frac{\partial}{\partial p_c} (L_{bd} \circ \varphi_H) \\ &\quad - L_{bc} \circ \varphi_H \cdot \frac{\partial}{\partial p_c} (L_{ad} \circ \varphi_H). \end{aligned}$$

Proof. Developing the following equalities

$$\begin{aligned} &\Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left[\tilde{\partial}_\alpha, \tilde{\partial}_\beta \right]_{(\rho, \eta) TE} \\ &= \left[\Gamma((\rho, \eta) T\varphi_L, \varphi_L) \tilde{\partial}_\alpha, \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \tilde{\partial}_\beta \right]_{(\rho, \eta) TE^*}, \\ &\Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left[\tilde{\partial}_\alpha, \overset{\cdot}{\tilde{\partial}}_b \right]_{(\rho, \eta) TE} \\ &= \left[\Gamma((\rho, \eta) T\varphi_L, \varphi_L) \tilde{\partial}_\alpha, \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \overset{\cdot}{\tilde{\partial}}_b \right]_{(\rho, \eta) TE^*} \end{aligned}$$

and

$$\begin{aligned} & \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left[\dot{\tilde{\partial}}_a, \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE} \\ &= \left[\Gamma((\rho, \eta) T\varphi_L, \varphi_L) \dot{\tilde{\partial}}_a, \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \dot{\tilde{\partial}}_b \right]_{(\rho, \eta)TE^*} \end{aligned}$$

it results the conclusion of the theorem.

q.e.d.

Corollary 4.1 *In particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain:*

$$(4.8)' \quad (L_{\alpha\beta}^\gamma \circ \pi) \circ \varphi_H = L_{\alpha\beta}^\gamma \circ \pi^*,$$

$$\begin{aligned} (4.9)' \quad [(L_{\alpha\beta}^\gamma \rho_\gamma^k) \circ \pi \cdot L_{kb}] \circ \varphi_H &= \rho_\alpha^i \circ \pi^* \cdot \frac{\partial}{\partial x^i} [(\rho_\beta^j \circ \pi \cdot L_{jb}) \circ \varphi_H] \\ &\quad - \rho_\beta^j \circ \pi^* \cdot \frac{\partial}{\partial x^j} [(\rho_\alpha^i \circ \pi \cdot L_{ib}) \circ \varphi_H] \\ &\quad + (\rho_\alpha^i \circ \pi \cdot L_{ia}) \circ \varphi_H \cdot \frac{\partial}{\partial p_a} [(\rho_\beta^j \circ \pi \cdot L_{jb}) \circ \varphi_H] \\ &\quad - (\rho_\beta^j \circ \pi \cdot L_{ja}) \circ \varphi_H \cdot \frac{\partial}{\partial p_a} [(\rho_\alpha^i \circ \pi \cdot L_{ib}) \circ \varphi_H], \end{aligned}$$

$$\begin{aligned} (4.10)' \quad 0 &= \rho_\alpha^i \circ \pi^* \cdot \frac{\partial}{\partial x^i} (L_{ba} \circ \varphi_H) \\ &\quad + (\rho_\alpha^i \circ \pi \cdot L_{bc}) \circ \varphi_H \cdot \frac{\partial}{\partial p_c} (L_{ba} \circ \varphi_H) \\ &\quad - L_{bc} \circ \varphi_H \cdot \frac{\partial}{\partial p_c} [(\rho_\alpha^i \circ \pi \cdot L_{ia}) \circ \varphi_H] \end{aligned}$$

and

$$\begin{aligned} (4.11)' \quad 0 &= L_{ac} \circ \varphi_H \cdot \frac{\partial}{\partial p_c} (L_{bd} \circ \varphi_H) \\ &\quad - L_{bc} \circ \varphi_H \cdot \frac{\partial}{\partial p_c} (L_{ad} \circ \varphi_H). \end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, we obtain:

$$\begin{aligned} (4.9)'' \quad 0 &= \frac{\partial}{\partial x^i} \left(\frac{\partial^2 L}{\partial x^j \partial y^k} \circ \varphi_H \right) - \frac{\partial}{\partial x^j} \left(\frac{\partial^2 L}{\partial x^i \partial y^k} \circ \varphi_H \right) \\ &\quad + \frac{\partial^2 L}{\partial x^i \partial y^h} \circ \varphi_H \cdot \frac{\partial}{\partial p_h} \left(\frac{\partial^2 L}{\partial x^j \partial y^k} \circ \varphi_H \right) - \frac{\partial^2 L}{\partial x^j \partial y^h} \circ \varphi_H \cdot \frac{\partial}{\partial p_h} \left(\frac{\partial^2 L}{\partial x^i \partial y^k} \circ \varphi_H \right) \end{aligned}$$

$$\begin{aligned} (4.10)'' \quad 0 &= \frac{\partial}{\partial x^i} \left(\frac{\partial^2 L}{\partial y^j \partial y^k} \circ \varphi_H \right) + \frac{\partial^2 L}{\partial x^i \partial y^h} \circ \varphi_H \cdot \frac{\partial}{\partial p_h} \left(\frac{\partial^2 L}{\partial y^j \partial y^k} \circ \varphi_H \right) \\ &\quad - \frac{\partial^2 L}{\partial x^j \partial y^h} \circ \varphi_H \cdot \frac{\partial}{\partial p_h} \left(\frac{\partial^2 L}{\partial x^i \partial y^k} \circ \varphi_H \right) \end{aligned}$$

and

$$\begin{aligned} (4.11)'' \quad 0 &= \frac{\partial^2 L}{\partial y^i \partial y^h} \circ \varphi_H \cdot \frac{\partial}{\partial p_h} \left(\frac{\partial^2 L}{\partial y^j \partial y^k} \circ \varphi_H \right) \\ &\quad - \frac{\partial^2 L}{\partial y^j \partial y^h} \circ \varphi_H \cdot \frac{\partial}{\partial p_h} \left(\frac{\partial^2 L}{\partial y^i \partial y^k} \circ \varphi_H \right). \end{aligned}$$

Theorem 4.2 *Dual, if the \mathbf{B}^V -morphism $((\rho, \eta) T\varphi_H, \varphi_H)$ is morphism of Lie algebroids, then we obtain:*

$$(4.12) \quad (L_{\alpha\beta}^\gamma \circ h \circ \pi^*) \circ \varphi_L = L_{\alpha\beta}^\gamma \circ h \circ \pi,$$

$$\begin{aligned}
(4.13) \quad \left[(L_{\alpha\beta}^\gamma \rho_\gamma^k) \circ h \circ \pi^* \cdot H_k^b \right] \circ \varphi_L &= \rho_\alpha^i \circ h \circ \pi \cdot \frac{\partial}{\partial x^i} \left[(\rho_\beta^j \circ h \circ \pi^* \cdot H_j^b) \circ \varphi_L \right] \\
&\quad - \rho_\beta^j \circ h \circ \pi \cdot \frac{\partial}{\partial x^j} \left[(\rho_\alpha^i \circ h \circ \pi^* \cdot H_i^b) \circ \varphi_L \right] \\
&\quad + (\rho_\alpha^i \circ h \circ \pi^* \cdot H_i^c) \circ \varphi_L \cdot \frac{\partial}{\partial y^c} \left[(\rho_\beta^j \circ h \circ \pi^* \cdot H_j^b) \circ \varphi_L \right] \\
&\quad - (\rho_\beta^j \circ h \circ \pi^* \cdot H_j^c) \circ \varphi_L \cdot \frac{\partial}{\partial y^c} \left[(\rho_\alpha^i \circ h \circ \pi^* \cdot H_i^b) \circ \varphi_L \right],
\end{aligned}$$

$$\begin{aligned}
(4.14) \quad 0 &= \rho_\alpha^i \circ h \circ \pi \cdot \frac{\partial}{\partial x^i} (H^{ba} \circ \varphi_L) \\
&\quad + (\rho_\alpha^i \circ h \circ \pi^* \cdot H^{bc}) \circ \varphi_L \frac{\partial}{\partial y^c} (H^{ba} \circ \varphi_L) \\
&\quad - H^{bc} \circ \varphi_L \cdot \frac{\partial}{\partial y^c} \left[(\rho_\alpha^i \circ h \circ \pi^* \cdot H_i^a) \circ \varphi_L \right]
\end{aligned}$$

and

$$\begin{aligned}
(4.15) \quad 0 &= H^{ac} \circ \varphi_L \cdot \frac{\partial}{\partial y^c} (H^{bd} \circ \varphi_L) \\
&\quad - H^{bc} \circ \varphi_L \cdot \frac{\partial}{\partial y^c} (H^{ad} \circ \varphi_L).
\end{aligned}$$

Proof. Developing the following equalities

$$\begin{aligned}
&\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \left[\tilde{\partial}_\alpha^*, \tilde{\partial}_\beta^* \right]_{(\rho, \eta) TE^*} \\
&= \left[\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \tilde{\partial}_\alpha^*, \Gamma((\rho, \eta) T\varphi_H, \varphi_H) \tilde{\partial}_\beta^* \right]_{(\rho, \eta) TE^*}, \\
&\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \left[\tilde{\partial}_\alpha^*, \tilde{\partial}^{\cdot b} \right]_{(\rho, \eta) TE^*} \\
&= \left[\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \tilde{\partial}_\alpha^*, \Gamma((\rho, \eta) T\varphi_H, \varphi_H) \tilde{\partial}^{\cdot b} \right]_{(\rho, \eta) TE^*}
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \left[\tilde{\partial}^{\cdot a}, \tilde{\partial}^{\cdot b} \right]_{(\rho, \eta) TE^*} \\
&= \left[\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \tilde{\partial}^{\cdot a}, \Gamma((\rho, \eta) T\varphi_H, \varphi_H) \tilde{\partial}^{\cdot b} \right]_{(\rho, \eta) TE^*}
\end{aligned}$$

it results the conclusion of the theorem.

q.e.d.

Corollary 4.2 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain:*

$$(4.12)' \quad \left(L_{\alpha\beta}^\gamma \circ \pi^* \right) \circ \varphi_L = L_{\alpha\beta}^\gamma \circ \pi,$$

$$\begin{aligned}
(4.13)' \quad \left[(L_{\alpha\beta}^\gamma \rho_\gamma^k) \circ \pi^* \cdot H_k^b \right] \circ \varphi_L &= \rho_\alpha^i \circ \pi \cdot \frac{\partial}{\partial x^i} \left[(\rho_\beta^j \circ \pi^* \cdot H_j^b) \circ \varphi_L \right] \\
&\quad - \rho_\beta^j \circ \pi \cdot \frac{\partial}{\partial x^j} \left[(\rho_\alpha^i \circ \pi^* \cdot H_i^b) \circ \varphi_L \right] \\
&\quad + (\rho_\alpha^i \circ \pi^* \cdot H_i^c) \circ \varphi_L \cdot \frac{\partial}{\partial y^c} \left[(\rho_\beta^j \circ \pi^* \cdot H_j^b) \circ \varphi_L \right] \\
&\quad - (\rho_\beta^j \circ \pi^* \cdot H_j^c) \circ \varphi_L \cdot \frac{\partial}{\partial y^c} \left[(\rho_\alpha^i \circ \pi^* \cdot H_i^b) \circ \varphi_L \right],
\end{aligned}$$

$$\begin{aligned}
(4.14)' \quad 0 &= \rho_\alpha^i \circ \pi \cdot \frac{\partial}{\partial x^i} (H^{ba} \circ \varphi_L) \\
&+ \left(\rho_\alpha^i \circ \pi^* \cdot H^{bc} \right) \circ \varphi_L \frac{\partial}{\partial y^c} (H^{ba} \circ \varphi_L) \\
&- H^{bc} \circ \varphi_L \cdot \frac{\partial}{\partial y^c} \left[\left(\rho_\alpha^i \circ \pi^* \cdot H_i^a \right) \circ \varphi_L \right]
\end{aligned}$$

and

$$\begin{aligned}
(4.15)' \quad 0 &= H^{ac} \circ \varphi_L \cdot \frac{\partial}{\partial y^c} (H^{bd} \circ \varphi_L) \\
&- H^{bc} \circ \varphi_L \cdot \frac{\partial}{\partial y^c} (H^{ad} \circ \varphi_L).
\end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, we obtain:

$$\begin{aligned}
(4.13)'' \quad 0 &= \frac{\partial}{\partial x^i} \left(\frac{\partial^2 H}{\partial x^k \partial p_j} \circ \varphi_L \right) - \frac{\partial}{\partial x^k} \left(\frac{\partial^2 H}{\partial x^i \partial p_j} \circ \varphi_L \right) \\
&+ \frac{\partial^2 H}{\partial x^i \partial p_h} \circ \varphi_L \cdot \frac{\partial}{\partial y^h} \left(\frac{\partial^2 H}{\partial x^k \partial p_j} \circ \varphi_L \right) - \frac{\partial^2 H}{\partial x^k \partial p_h} \circ \varphi_L \cdot \frac{\partial}{\partial y^h} \left(\left(\frac{\partial^2 H}{\partial x^i \partial p_j} \circ \varphi_L \right) \circ \varphi_L \right)
\end{aligned}$$

$$\begin{aligned}
(4.14)'' \quad 0 &= \frac{\partial}{\partial x^k} \left(\frac{\partial^2 H}{\partial p_i \partial p_j} \circ \varphi_L \right) + \frac{\partial^2 H}{\partial p_i \partial p_h} \circ \varphi_L \cdot \frac{\partial}{\partial y^h} \left(\frac{\partial^2 H}{\partial x^k \partial p_j} \circ \varphi_L \right) \\
&- \frac{\partial^2 H}{\partial p_j \partial p_h} \circ \varphi_L \cdot \frac{\partial}{\partial y^h} \left(\frac{\partial^2 H}{\partial x^k \partial p_i} \circ \varphi_L \right)
\end{aligned}$$

and

$$\begin{aligned}
(4.15)'' \quad 0 &= \frac{\partial^2 H}{\partial p_i \partial p_k} \circ \varphi_L \cdot \frac{\partial}{\partial y^k} \left(\frac{\partial^2 H}{\partial p_j \partial p_h} \circ \varphi_L \right) \\
&- \frac{\partial^2 H}{\partial p_j \partial p_k} \circ \varphi_L \cdot \frac{\partial}{\partial y^k} \left(\frac{\partial^2 H}{\partial p_i \partial p_h} \circ \varphi_L \right).
\end{aligned}$$

Definition 4.1 If $((\rho, \eta) T\varphi_L, \varphi_L)$ and $((\rho, \eta) T\varphi_H, \varphi_H)$ are Lie algebroids morphisms, then we will say that (E, π, M) and $\left(E, \pi^*, M \right)$ are Legendre (ρ, η, h) -equivalent and we will write

$$(E, \pi, M) \xrightarrow[\text{(\rho, \eta, h)}]{\mathcal{L}} \left(E, \pi^*, M \right).$$

5 Duality between adapted (ρ, η) -basis

If $(\rho, \eta) \Gamma$ is a (ρ, η) -connection for the vector bundle (E, π, M) , then

$$(5.1) \quad \left(\tilde{\partial}_\alpha - (\rho, \eta) \Gamma_\alpha^a \dot{\tilde{\partial}}_a, \dot{\tilde{\partial}}_a \right) = \left(\tilde{\tilde{\partial}}_\alpha, \dot{\tilde{\tilde{\partial}}}_a \right).$$

is the adapted (ρ, η) -base of $(\Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot)$. (see [1, 4])

If $(\rho, \eta) \Gamma$ is a (ρ, η) -connection for the vector bundle $\left(E, \pi^*, M \right)$, then

$$(5.1)' \quad \left(\tilde{\tilde{\partial}}_\alpha + (\rho, \eta) \Gamma_{b\alpha}^{\cdot b} \dot{\tilde{\tilde{\partial}}}, \dot{\tilde{\tilde{\partial}}}^{\cdot a} \right) = \left(\tilde{\tilde{\tilde{\partial}}}_\alpha, \dot{\tilde{\tilde{\tilde{\partial}}}}^{\cdot a} \right).$$

is the adapted (ρ, η) -base of $\left(\Gamma \left((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E \right), +, \cdot \right)$. (see [1, 5])

Definition 5.1 If

$$(E, \pi, M) \xrightarrow[\text{(\rho, \eta, h)}]{\mathcal{L}} \left(E, \pi^*, M \right).$$

and

$$\Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left(\tilde{\delta}_\alpha \right) = \tilde{\delta}_\alpha^*,$$

$$\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \left(\tilde{\delta}_\alpha \right) = \tilde{\delta}_\alpha^*,$$

then we will say that (E, π, M) and $\left(\overset{*}{E}, \overset{*}{\pi}, M \right)$ are horizontal Legendre (ρ, η, h) -equivalent and we will write

$$(E, \pi, M) \xrightarrow[\substack{\mathcal{HL}}]{(\rho, \eta, h)} \left(\overset{*}{E}, \overset{*}{\pi}, M \right).$$

Theorem 5.1 *If*

$$(E, \pi, M) \xrightarrow[\substack{\mathcal{HL}}]{(\rho, \eta, h)} \left(\overset{*}{E}, \overset{*}{\pi}, M \right),$$

then we obtain:

$$(5.2) \quad (\rho, \eta) \Gamma_{b\alpha} = \left[(\rho_\alpha^i \circ h \circ \pi) \cdot L_{ib} - (\rho, \eta) \Gamma_\alpha^a \cdot L_{ab} \right] \circ \varphi_H$$

and

$$(5.3) \quad -(\rho, \eta) \Gamma_\alpha^a = \left[\left(\rho_\alpha^i \circ h \circ \pi \right) \cdot H_i^a + (\rho, \eta) \Gamma_{b\alpha} \cdot H^{ba} \right] \circ \varphi_L.$$

Corollary 5.1 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$(5.2)' \quad \rho \Gamma_{b\alpha} = \left[(\rho_\alpha^i \circ \pi) \cdot L_{ib} - \rho \Gamma_\alpha^a \cdot L_{ab} \right] \circ \varphi_H$$

and

$$(5.3)' \quad -\rho \Gamma_\alpha^a = \left[\left(\rho_\alpha^i \circ \pi \right) \cdot H_i^a + \rho \Gamma_{b\alpha} \cdot H^{ba} \right] \circ \varphi_L.$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, we obtain the equality implies the equality

$$(5.2)'' \quad \Gamma_{jk} = \left[\frac{\partial^2 L}{\partial x^i \partial y^j} - \Gamma_k^i \frac{\partial^2 L}{\partial y^i \partial y^j} \right] \circ \varphi_H$$

and

$$(5.3)'' \quad -\Gamma_k^i = \left[\frac{\partial^2 H}{\partial x^k \partial p_i} + \Gamma_{jk} \frac{\partial^2 H}{\partial p_j \partial p_i} \right] \circ \varphi_L.$$

If the Lagrangian L is regular, then we will define the real local functions \tilde{L}^{ab} such that

$$\left\| \tilde{L}^{ab}(u_x) \right\| = \|L_{ab}(u_x)\|^{-1}, \quad \forall u_x \in \pi^{-1}(U).$$

If the Hamiltonian H is regular, then we will define the real local functions \tilde{H}_{ab} such that

$$\left\| \tilde{H}_{ab} \left(\overset{*}{u}_x \right) \right\| = \left\| H^{ab} \left(\overset{*}{u}_x \right) \right\|^{-1}, \quad \forall \overset{*}{u}_x \in \overset{*}{\pi}^{-1}(U).$$

Remark 5.1 If the Lagrangian L is regular and

$$(E, \pi, M) \xrightarrow[\substack{\mathcal{HL}}]{(\rho, \eta, h)} \left(\overset{*}{E}, \overset{*}{\pi}, M \right)$$

then, the Hamiltonian H is regular,

$$(5.4) \quad \tilde{H}_{ab} = L_{ab} \circ \varphi_H.$$

and

$$(5.5) \quad [(\rho_\alpha^i \circ h \circ \pi) \cdot L_{ia}] \circ \varphi_H = - \left(\rho_\alpha^i \circ h \circ \pi^* \right) \cdot H_i^b \cdot \tilde{H}_{ba}.$$

It is known that the following equalities hold good

$$(5.6) \quad [\tilde{\delta}_\alpha, \tilde{\delta}_\beta]_{(\rho, \eta)TE} = \left(L_{\alpha\beta}^\gamma \circ h \circ \pi \right) \tilde{\delta}_\gamma + (\rho, \eta, h) \mathbb{R}^a_{\alpha\beta} \dot{\tilde{\partial}}^b_a,$$

and

$$(5.6)' \quad [\tilde{\delta}_\alpha^*, \tilde{\delta}_\beta^*]_{(\rho, \eta)TE^*} = \left(L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) \tilde{\delta}_\gamma^* + (\rho, \eta, h) \mathbb{R}^a_{\alpha\beta} \dot{\tilde{\partial}}^b_a,$$

Theorem 5.2 *If*

$$(E, \pi, M) \xrightarrow{(\rho, \eta, h)} \left(E^*, \pi^*, M \right),$$

then, we obtain:

$$(5.7) \quad (\rho, \eta, h) \mathbb{R}_{b\alpha\beta} = \left[(\rho, \eta, h) \mathbb{R}^a_{\alpha\beta} \cdot L_{ab} \right] \circ \varphi_H$$

and

$$(5.8) \quad (\rho, \eta, h) \mathbb{R}^a_{\alpha\beta} = \left[(\rho, \eta, h) \mathbb{R}_{b\alpha\beta} \cdot H^{ba} \right] \circ \varphi_L.$$

Corollary 5.2 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$(5.7)' \quad \rho \mathbb{R}_{b\alpha\beta} = \left(\rho \mathbb{R}^a_{\alpha\beta} L_{ab} \right) \circ \varphi_H$$

and

$$(5.8)' \quad \rho \mathbb{R}^a_{\alpha\beta} = \left(\rho \mathbb{R}_{b\alpha\beta} H^{ba} \right) \circ \varphi_L.$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, we obtain

$$(5.7)'' \quad \mathbb{R}_{j\ hk} = \left(\mathbb{R}^i_{\ hk} \cdot \frac{\partial^2 L}{\partial y^i \partial y^j} \right) \circ \varphi_H$$

and

$$(5.8)'' \quad \mathbb{R}^i_{\ hk} = \left(\mathbb{R}_{j\ hk} \cdot \frac{\partial^2 H}{\partial p_j \partial p_i} \right) \circ \varphi_L.$$

Theorem 5.3 *If*

$$(E, \pi, M) \xrightarrow{(\rho, \eta, h)} \left(E^*, \pi^*, M \right),$$

then we obtain

$$\begin{aligned}
(5.9) \quad \left(\frac{\partial(\rho, \eta) \Gamma_{\alpha}^a}{\partial y^b} \cdot L_{ac} \right) \circ \varphi_H &= L_{ba} \circ \varphi_H \cdot \frac{\partial(\rho, \eta) \Gamma_{c\alpha}}{\partial p_a} \\
&+ \left(\rho_{\alpha}^i \circ h \circ \pi^* \right) \cdot \frac{\partial}{\partial x^i} (L_{bc} \circ \varphi_H) \\
&+ (\rho, \eta) \Gamma_{a\alpha} \cdot \frac{\partial}{\partial p_a} (L_{bc} \circ \varphi_H)
\end{aligned}$$

and

$$\begin{aligned}
(5.10) \quad - \left(\frac{\partial(\rho, \eta) \Gamma_{b\alpha}}{\partial p_a} \cdot H^{bc} \right) \circ \varphi_L &= H^{ba} \circ \varphi_L \cdot \frac{\partial(\rho, \eta) \Gamma_{\alpha}^c}{\partial y^a} \\
&+ \left(\rho_{\alpha}^i \circ h \circ \pi \right) \cdot \frac{\partial}{\partial x^i} (H^{bc} \circ \varphi_L) \\
&+ (\rho, \eta) \Gamma_{\alpha}^a \cdot \frac{\partial}{\partial y^a} (H^{bc} \circ \varphi_L)
\end{aligned}$$

Proof. Developing the following equalities

$$\begin{aligned}
&\Gamma((\rho, \eta) T\varphi_L, \varphi_L) \left(\left[\tilde{\delta}_{\alpha}, \dot{\tilde{\partial}}_a \right]_{(\rho, \eta) TE} \right) \\
&= \left[\Gamma((\rho, \eta) T\varphi_L, \varphi_L) \tilde{\delta}_{\alpha}, \Gamma((\rho, \eta) T\varphi_L, \varphi_L) \dot{\tilde{\partial}}_a \right]_{(\rho, \eta) TE}
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \left(\left[\tilde{\delta}_{\alpha}^*, \dot{\tilde{\partial}}^a \right]_{(\rho, \eta) TE} \right) \\
&= \left[\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \tilde{\delta}_{\alpha}^*, \Gamma((\rho, \eta) T\varphi_H, \varphi_H) \dot{\tilde{\partial}}^a \right]_{(\rho, \eta) TE}
\end{aligned}$$

it results the conclusion of the theorem.

q.e.d.

Corollary 5.2 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$\begin{aligned}
(5.9)' \quad \left(\frac{\partial \rho \Gamma_{\alpha}^a}{\partial y^b} \cdot L_{ac} \right) \circ \varphi_H &= L_{ba} \circ \varphi_H \cdot \frac{\partial \rho \Gamma_{c\alpha}}{\partial p_a} \\
&+ \left(\rho_{\alpha}^i \circ \pi^* \right) \cdot \frac{\partial}{\partial x^i} (L_{bc} \circ \varphi_H) \\
&+ \rho \Gamma_{a\alpha} \cdot \frac{\partial}{\partial p_a} (L_{bc} \circ \varphi_H)
\end{aligned}$$

and

$$\begin{aligned}
(5.10)' \quad - \left(\frac{\partial \rho \Gamma_{b\alpha}}{\partial p_a} \cdot H^{bc} \right) \circ \varphi_L &= H^{ba} \circ \varphi_L \cdot \frac{\partial \rho \Gamma_{\alpha}^c}{\partial y^a} \\
&+ \left(\rho_{\alpha}^i \circ \pi \right) \cdot \frac{\partial}{\partial x^i} (H^{bc} \circ \varphi_L) \\
&+ \rho \Gamma_{\alpha}^a \cdot \frac{\partial}{\partial y^a} (H^{bc} \circ \varphi_L)
\end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, we obtain

$$\begin{aligned}
(5.9)'' \quad \left(\frac{\partial \Gamma_k^i}{\partial y^j} \cdot \frac{\partial^2 L}{\partial y^i \partial y^h} \right) \circ \varphi_H &= \frac{\partial^2 L}{\partial y^j \partial y^i} \circ \varphi_H \cdot \frac{\partial \rho \Gamma_{hk}}{\partial p_i} \\
&+ \frac{\partial}{\partial x^k} \left(\frac{\partial^2 L}{\partial y^j \partial y^h} \circ \varphi_H \right) \\
&+ \Gamma_{ik} \cdot \frac{\partial}{\partial p_i} \left(\frac{\partial^2 L}{\partial y^j \partial y^h} \circ \varphi_H \right)
\end{aligned}$$

and

$$(5.10)'' \quad - \left(\frac{\partial \Gamma_{jk}}{\partial p_i} \cdot \frac{\partial^2 H}{\partial p_j \partial p_h} \right) \circ \varphi_L = \frac{\partial^2 H}{\partial p_i \partial p_e} \circ \varphi_L \cdot \frac{\partial \rho \Gamma_k^h}{\partial y^e} + \frac{\partial}{\partial x^k} \left(\frac{\partial^2 H}{\partial p_j \partial p_h} \circ \varphi_L \right) + \Gamma_k^i \cdot \frac{\partial}{\partial y^i} \left(\frac{\partial^2 H}{\partial p_j \partial p_h} \circ \varphi_L \right)$$

The dual natural (ρ, η) -base of the natural (ρ, η) -base $\left(\tilde{\partial}_\alpha, \tilde{\partial}_a \right)$ is denoted $(d\tilde{z}^\alpha, d\tilde{y}^a)$ and the dual adapted (ρ, η) -base of the adapted (ρ, η) -base $\left(\tilde{\delta}_\alpha, \tilde{\delta}_a \right)$ is denoted

$$(5.11) \quad (d\tilde{z}^\alpha, d\tilde{y}^a) \stackrel{put}{=} (d\tilde{z}^\alpha, (\rho, \eta) \Gamma_\alpha^a \cdot d\tilde{z}^\alpha + d\tilde{y}^a).$$

The dual natural (ρ, η) -base of the natural (ρ, η) -base $\left(\tilde{\partial}_\alpha^*, \tilde{\partial}_a^* \right)$ is denoted $(d\tilde{z}^\alpha, d\tilde{p}_a)$ and the dual adapted (ρ, η) -base of the adapted (ρ, η) -base $\left(\tilde{\delta}_\alpha^*, \tilde{\delta}_a^* \right)$ is denoted

$$(5.11)' \quad (d\tilde{z}^\alpha, d\tilde{p}_a) \stackrel{put}{=} (d\tilde{z}^\alpha, -(\rho, \eta) \Gamma_{a\alpha} \cdot d\tilde{z}^\alpha + d\tilde{p}_a).$$

Let

$$(\Lambda((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot, \wedge)$$

be exterior differential $\mathcal{F}(E)$ -algebra of the generalized tangent bundle $((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$ and let

$$\left(\Lambda \left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right), +, \cdot, \wedge \right).$$

be exterior differential $\mathcal{F} \left(E^* \right)$ -algebra of the generalized tangent bundle $\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$.

Using the \mathbf{B}^v -morphism (4.6) given by the equalities (4.7), we obtain the application

$$\begin{aligned} \Lambda \left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) & \xrightarrow{((\rho, \eta) T\varphi_L, \varphi_L)^*} \Lambda((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \\ \Lambda^q \left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) \ni \omega & \longmapsto ((\rho, \eta) T\varphi_L, \varphi_L)^*(\omega) \end{aligned}$$

where

$$((\rho, \eta) T\varphi_L, \varphi_L)^*(\omega)(X_1, \dots, X_q) = \omega(\Gamma((\rho, \eta) T\varphi_L, \varphi_L) X_1, \dots, \Gamma((\rho, \eta) T\varphi_L, \varphi_L) X_q) \circ \varphi_L,$$

for any $X_1, \dots, X_q \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$.

Using the \mathbf{B}^v -morphism (4.6)' given by the equalities (4.7)', we obtain the application

$$\begin{aligned} \Lambda((\rho, \eta) TE, (\rho, \eta) \tau_E, E) & \xrightarrow{((\rho, \eta) T\varphi_H, \varphi_H)^*} \Lambda \left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) \\ \Lambda^q((\rho, \eta) TE, (\rho, \eta) \tau_E, E) \ni \omega & \longmapsto ((\rho, \eta) T\varphi_H, \varphi_H)^*(\omega) \end{aligned}$$

where

$$((\rho, \eta) T\varphi_H, \varphi_H)^*(\omega)(X_1, \dots, X_q) = \omega(\Gamma((\rho, \eta) T\varphi_H, \varphi_H) X_1, \dots, \Gamma((\rho, \eta) T\varphi_H, \varphi_H) X_q) \circ \varphi_H,$$

for any $X_1, \dots, X_q \in \Gamma \left((\rho, \eta) T E^*, (\rho, \eta) \tau_E^*, E \right)$.

Theorem 5.4 *The equality*

$$\Gamma \left((\rho, \eta) T \varphi_L, \varphi_L \right) \left(\tilde{\delta}_\alpha \right) = \tilde{\delta}_\alpha^*$$

is equivalent with the equality:

$$(5.12) \quad ((\rho, \eta) T \varphi_L, \varphi_L)^* (\delta \tilde{p}_a) = L_{ab} \cdot \delta \tilde{y}^b.$$

Dual, the equality

$$\Gamma \left((\rho, \eta) T \varphi_H, \varphi_H \right) \left(\tilde{\delta}_\alpha^* \right) = \tilde{\delta}_\alpha,$$

is equivalent with the equality:

$$(5.12)' \quad \Gamma \left((\rho, \eta) T \varphi_H, \varphi_H \right)^* (\delta \tilde{y}^a) = H^{ab} \cdot \delta \tilde{p}_b$$

6 Duality between distinguished linear (ρ, η) -connections

Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle (E, π, M) and let

$$(6.1) \quad (X, T) \xrightarrow{(\rho, \eta) D} (\rho, \eta) D_X T$$

be a covariant (ρ, η) -derivative for the tensor algebra of generalized tangent bundle

$$((\rho, \eta) T E, (\rho, \eta) \tau_E, E)$$

which preserves the horizontal and vertical *IDS* by parallelism.

If (U, s_U) is a vector local $(m + r)$ -chart for (E, π, M) , then the real local functions

$$((\rho, \eta) H_{\beta\gamma}^\alpha, (\rho, \eta) H_{b\gamma}^a, (\rho, \eta) V_{\beta c}^\alpha, (\rho, \eta) V_{bc}^a)$$

defined on $\pi^{-1}(U)$ and determined by the following equalities:

$$(6.2) \quad \begin{aligned} (\rho, \eta) D_{\tilde{\delta}_\gamma} \tilde{\delta}_\beta &= (\rho, \eta) H_{\beta\gamma}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta) D_{\tilde{\delta}_\gamma} \tilde{\delta}_b &= (\rho, \eta) H_{b\gamma}^a \tilde{\delta}_a \\ (\rho, \eta) D_{\tilde{\delta}_c} \tilde{\delta}_\beta &= (\rho, \eta) V_{\beta c}^\alpha \tilde{\delta}_\alpha, & (\rho, \eta) D_{\tilde{\delta}_c} \tilde{\delta}_b &= (\rho, \eta) V_{bc}^a \tilde{\delta}_a \end{aligned}$$

are the components of a distinguished linear (ρ, η) -connection $((\rho, \eta) H, (\rho, \eta) V)$.

Let $(\rho, \eta) \Gamma$ be a (ρ, η) -connection for the vector bundle $\left(E^*, \pi^*, M \right)$ and let

$$(6.1)' \quad (X, T) \xrightarrow{(\rho, \eta) D^*} (\rho, \eta) D_X^* T$$

be a covariant (ρ, η) -derivative for the tensor algebra of generalized tangent bundle

$$\left((\rho, \eta) T E^*, (\rho, \eta) \tau_E^*, E^* \right)$$

which preserves the horizontal and vertical *IDS* by parallelism.

If (U, s_U^*) is a vector local $(m+r)$ -chart for (E, π, M) , then the real local functions

$$\left((\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha}, (\rho, \eta) \overset{*}{H}_{b\gamma}^a, (\rho, \eta) \overset{*}{V}_{\beta}^{\alpha c}, (\rho, \eta) \overset{*}{V}_b^{ac} \right)$$

defined on $\pi^{*-1}(U)$ and determined by the following equalities:

$$(6.2)' \quad \begin{aligned} (\rho, \eta) \overset{*}{D}_{\tilde{\delta}_\gamma}^* \tilde{\delta}_\beta^{\alpha} &= (\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha} \tilde{\delta}_\alpha^{\alpha}, & (\rho, \eta) \overset{*}{D}_{\tilde{\delta}_\gamma}^* \tilde{\delta}^{\cdot a} &= (\rho, \eta) \overset{*}{H}_{b\gamma}^a \tilde{\delta}^{\cdot b} \\ (\rho, \eta) \overset{*}{D}_{\tilde{\delta}}^* \tilde{\delta}_\beta^{\alpha c} &= (\rho, \eta) \overset{*}{V}_{\beta}^{\alpha c} \tilde{\delta}_\alpha^{\alpha}, & (\rho, \eta) \overset{*}{D}_{\tilde{\delta}}^* \tilde{\delta}^{\cdot bc} &= (\rho, \eta) \overset{*}{V}_a^{bc} \tilde{\delta}^{\cdot a} \end{aligned}$$

are the components of a distinguished linear (ρ, η) -connection

$$\left((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right).$$

Theorem 6.1 *If*

$$(E, \pi, M) \xrightarrow[\text{(\rho, \eta, h)}]{\mathcal{HL}} \left(\overset{*}{E}, \overset{*}{\pi}, M \right)$$

and

$$\Gamma((\rho, \eta) T\varphi_L, \varphi_L)((\rho, \eta) D_X Y) = (\rho, \eta) \overset{*}{D}_{\Gamma((\rho, \eta) T\varphi_L, \varphi_L)X} \Gamma((\rho, \eta) T\varphi_L, \varphi_L) Y,$$

for any $X, Y \in \Gamma((\rho, \eta) TE, (\rho, \eta) \tau_E, E)$, then we obtain:

$$(6.3) \quad (\rho, \eta) H_{\beta\gamma}^{\alpha} \circ \varphi_H = (\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha},$$

$$(6.4) \quad \begin{aligned} \left((\rho, \eta) H_{b\gamma}^a \cdot L_{ac} \right) \circ \varphi_H &= \left(\rho_\gamma^k \circ h \circ \pi^* \right) \cdot \frac{\partial}{\partial x^k} (L_{bc} \circ \varphi_H) \\ &+ (\rho, \eta) \Gamma_{b\gamma} \cdot \frac{\partial}{\partial p_b} (L_{bc} \circ \varphi_H) \\ &- (\rho, \eta) \overset{*}{H}_{b\gamma}^a \cdot (L_{ac} \circ \varphi_H), \end{aligned}$$

$$(6.5) \quad (\rho, \eta) V_{\beta d}^{\alpha} \circ \varphi_H = (\rho, \eta) \overset{*}{V}_{\beta}^{\alpha c} \cdot (L_{cd} \circ \varphi_H)$$

and

$$(6.6) \quad \begin{aligned} ((\rho, \eta) V_{bc}^a \cdot L_{ad}) \circ \varphi_H &= (L_{ce} \circ \varphi_H) \cdot \frac{\partial}{\partial p_e} (L_{bd} \circ \varphi_H) \\ &- (L_{ce} \circ \varphi_H) \cdot (\rho, \eta) \overset{*}{V}_d^{ef} \cdot (L_{bf} \circ \varphi_H). \end{aligned}$$

Corollary 6.1 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$(6.3)' \quad \rho H_{\beta\gamma}^{\alpha} \circ \varphi_H = \rho \overset{*}{H}_{\beta\gamma}^{\alpha},$$

$$(6.4)' \quad \begin{aligned} \left(\rho H_{b\gamma}^a \cdot L_{ac} \right) \circ \varphi_H &= \left(\rho_\gamma^k \circ \pi^* \right) \cdot \frac{\partial}{\partial x^k} (L_{bc} \circ \varphi_H) \\ &+ \rho \Gamma_{b\gamma} \cdot \frac{\partial}{\partial p_b} (L_{bc} \circ \varphi_H) \\ &- \rho \overset{*}{H}_{b\gamma}^a \cdot (L_{ac} \circ \varphi_H), \end{aligned}$$

$$(6.5)' \quad \rho V_{\beta d}^\alpha \circ \varphi_H = \rho V_\beta^{*\alpha c} \cdot (L_{cd} \circ \varphi_H)$$

and

$$(6.6)' \quad \begin{aligned} (\rho V_{bc}^a \cdot L_{ad}) \circ \varphi_H &= (L_{ce} \circ \varphi_H) \cdot \frac{\partial}{\partial p_e} (L_{bd} \circ \varphi_H) \\ &\quad - (L_{ce} \circ \varphi_H) \cdot \rho V_d^{*ef} \cdot (L_{bf} \circ \varphi_H). \end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, we obtain

$$(6.3)'' \quad H_{jk}^i \circ \varphi_H = H_{jk}^{*i},$$

$$(6.4)'' \quad \begin{aligned} \left(H_{jk}^i \cdot \frac{\partial^2 L}{\partial y^i \partial y^h} \right) \circ \varphi_H &= \frac{\partial}{\partial x^k} \left(\frac{\partial^2 L}{\partial y^j \partial y^h} \circ \varphi_H \right) \\ &\quad + \Gamma_{jk} \cdot \frac{\partial}{\partial p_e} \left(\frac{\partial^2 L}{\partial y^e \partial y^h} \circ \varphi_H \right) \\ &\quad - H_{jk}^{*i} \cdot \left(\frac{\partial^2 L}{\partial y^i \partial y^h} \circ \varphi_H \right), \end{aligned}$$

$$(6.5)'' \quad V_{jk}^i \circ \varphi_H = \rho V_j^{*ih} \cdot \left(\frac{\partial^2 L}{\partial y^h \partial y^k} \circ \varphi_H \right)$$

and

$$(6.6)'' \quad \begin{aligned} \left(V_{jk}^i \cdot \frac{\partial^2 L}{\partial y^i \partial y^h} \right) \circ \varphi_H &= \left(\frac{\partial^2 L}{\partial y^k \partial y^e} \circ \varphi_H \right) \cdot \frac{\partial}{\partial p_e} \left(\frac{\partial^2 L}{\partial y^j \partial y^h} \circ \varphi_H \right) \\ &\quad - \left(\frac{\partial^2 L}{\partial y^k \partial y^e} \circ \varphi_H \right) \cdot \rho V_h^{*ef} \cdot \left(\frac{\partial^2 L}{\partial y^j \partial y^f} \circ \varphi_H \right). \end{aligned}$$

Theorem 6.2 *Dual, if*

$$(E, \pi, M) \xrightarrow{(\rho, \eta, h)} \left(E^*, \pi^*, M \right)$$

and

$$\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \left((\rho, \eta) \overset{*}{D}_X Y \right) = (\rho, \eta) D_{\Gamma((\rho, \eta) T\varphi_H, \varphi_H) X} \Gamma((\rho, \eta) T\varphi_H, \varphi_H) Y,$$

for any $X, Y \in \Gamma \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}^*, \overset{*}{E} \right)$, then we obtain

$$(6.7) \quad (\rho, \eta) H_{\beta\gamma}^{*\alpha} \circ \varphi_L = (\rho, \eta) H_{\beta\gamma}^\alpha,$$

$$(6.8) \quad \begin{aligned} \left((\rho, \eta) \overset{*a}{H}_{b\gamma} \cdot H^{bc} \right) \circ \varphi_L &= (\rho_\gamma^k \circ h \circ \pi) \cdot \frac{\partial}{\partial x^k} (H^{ac} \circ \varphi_L) \\ &\quad + (\rho, \eta) \Gamma_\gamma^b \cdot \frac{\partial}{\partial y^b} (H^{ac} \circ \varphi_L) \\ &\quad - (\rho, \eta) H_{b\gamma}^a \cdot (H^{bc} \circ \varphi_L), \end{aligned}$$

$$(6.9) \quad (\rho, \eta) \overset{*}{V}_\beta^{\alpha c} \circ \varphi_L = (\rho, \eta) V_{\beta c}^\alpha \cdot (H^{cd} \circ \varphi_L)$$

and

$$(6.10) \quad \begin{aligned} \left((\rho, \eta) V_a^{*bc} \cdot H^{ad} \right) \circ \varphi_L &= (H^{ce} \circ \varphi_H) \cdot \frac{\partial}{\partial y^e} (H^{bd} \circ \varphi_L) \\ &\quad - (H^{ce} \circ \varphi_L) \cdot (\rho, \eta) V_{ef}^d \cdot (H^{bf} \circ \varphi_L). \end{aligned}$$

Corollary 6.1 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$(6.7)' \quad \rho H_{\beta\gamma}^{*\alpha} \circ \varphi_L = \rho H_{\beta\gamma}^\alpha,$$

$$(6.8)' \quad \begin{aligned} \left(\rho H_{b\gamma}^{*a} \cdot H^{bc} \right) \circ \varphi_L &= (\rho_\gamma^k \circ \pi) \cdot \frac{\partial}{\partial x^k} (H^{ac} \circ \varphi_L) \\ &\quad + \rho \Gamma_\gamma^b \cdot \frac{\partial}{\partial y^b} (H^{ac} \circ \varphi_L) \\ &\quad - \rho H_{b\gamma}^a \cdot (H^{bc} \circ \varphi_L), \end{aligned}$$

$$(6.9)' \quad \rho V_\beta^{*\alpha c} \circ \varphi_L = \rho V_{\beta c}^\alpha \cdot (H^{cd} \circ \varphi_L)$$

and

$$(6.10)' \quad \begin{aligned} \left(\rho V_a^{*bc} \cdot H^{ad} \right) \circ \varphi_L &= (H^{ce} \circ \varphi_H) \cdot \frac{\partial}{\partial y^e} (H^{bd} \circ \varphi_L) \\ &\quad - (H^{ce} \circ \varphi_L) \cdot (\rho, \eta) V_{ef}^d \cdot (H^{bf} \circ \varphi_L). \end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, we obtain

$$(6.7)' \quad H_{jk}^{*i} \circ \varphi_L = H_{jk}^i,$$

$$(6.8)' \quad \begin{aligned} \left(H_{jk}^{*i} \cdot \frac{\partial^2 H}{\partial p_j \partial p_h} \right) \circ \varphi_L &= \frac{\partial}{\partial x^k} \left(\frac{\partial^2 H}{\partial p_i \partial p_h} \circ \varphi_L \right) \\ &\quad + \Gamma_k^e \cdot \frac{\partial}{\partial y^e} \left(\frac{\partial^2 H}{\partial p_i \partial p_h} \circ \varphi_L \right) \\ &\quad - \rho H_{jk}^i \cdot \left(\frac{\partial^2 H}{\partial p_j \partial p_h} \circ \varphi_L \right), \end{aligned}$$

$$(6.9)' \quad V_j^{*ik} \circ \varphi_L = \rho V_{jh}^i \cdot \left(\frac{\partial^2 H}{\partial p_h \partial p_k} \circ \varphi_L \right)$$

and

$$(6.10)' \quad \begin{aligned} \left(\rho V_i^{*jk} \cdot \frac{\partial^2 H}{\partial p_i \partial p_h} \right) \circ \varphi_L &= \left(\frac{\partial^2 H}{\partial p_k \partial p_e} \circ \varphi_H \right) \cdot \frac{\partial}{\partial y^e} \left(\frac{\partial^2 H}{\partial p_j \partial p_h} \circ \varphi_L \right) \\ &\quad - \left(\frac{\partial^2 H}{\partial p_k \partial p_e} \circ \varphi_L \right) \cdot V_{ef}^h \cdot \left(\frac{\partial^2 H}{\partial p_j \partial p_f} \circ \varphi_L \right). \end{aligned}$$

7 Duality between mechanical (ρ, η) -systems

Let $((E, \pi, M), F_e, (\rho, \eta)\Gamma)$ be a mechanical (ρ, η) -system.

If $g \in \mathbf{Man}(E, E)$ such that (g, h) is a \mathbf{B}^V -morphism locally invertible of (E, π, M) source and (E, π, M) target, on components g_b^a , then the **Mod**-endomorphism

$$(7.1) \quad \begin{aligned} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) &\xrightarrow{\mathcal{J}_{(g, h)}} \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E) \\ \tilde{Z}^a \tilde{\partial}_a + Y^b \dot{\tilde{\partial}}_b &\longmapsto (\tilde{g}_a^b \circ h \circ \pi) \tilde{Z}^a \dot{\tilde{\partial}}_b \end{aligned}$$

is the almost tangent structure associated to the \mathbf{B}^V -morphism (g, h) .

The vertical section

$$(7.2) \quad \mathbb{C} = y^a \dot{\tilde{\partial}}_a$$

is the Liouville section.

Let $\left(\left(\begin{smallmatrix} * \\ E, \pi, M \end{smallmatrix}\right), F_e, (\rho, \eta)\Gamma\right)$ be a dual mechanical (ρ, η) -system.

If $g \in \mathbf{Man}\left(\begin{smallmatrix} * \\ E, E \end{smallmatrix}\right)$ be such that (g, h) is a \mathbf{B}^V -morphism locally invertible of $\left(\begin{smallmatrix} * \\ E, \pi, M \end{smallmatrix}\right)$ source and (E, π, M) target, on components g^{ab} , then the **Mod**-endomorphism

$$(7.1)' \quad \begin{aligned} \Gamma\left((\rho, \eta)T\begin{smallmatrix} * \\ E \end{smallmatrix}, (\rho, \eta)\tau_{\begin{smallmatrix} * \\ E \end{smallmatrix}}, \begin{smallmatrix} * \\ E \end{smallmatrix}\right) &\xrightarrow{\mathcal{J}_{(g, h)}} \Gamma\left((\rho, \eta)T\begin{smallmatrix} * \\ E \end{smallmatrix}, (\rho, \eta)\tau_{\begin{smallmatrix} * \\ E \end{smallmatrix}}, \begin{smallmatrix} * \\ E \end{smallmatrix}\right) \\ \tilde{Z}^a \dot{\tilde{\partial}}_a + Y_b \dot{\tilde{\partial}}^b &\longmapsto (\tilde{g}_{ba} \circ h \circ \pi^*) \tilde{Z}^a \dot{\tilde{\partial}}^b \end{aligned}$$

is the almost tangent structure associated to the \mathbf{B}^V -morphism (g, h) .

The vertical section

$$(7.2)' \quad \mathbb{C} = p_b \dot{\tilde{\partial}}^b$$

is the Liouville section.

Let

$$(7.3) \quad S = y^b (g_b^a \circ h \circ \pi) \tilde{\partial}_a - 2 \left(G^a - \frac{1}{4} F^a\right) \dot{\tilde{\partial}}_a$$

be the (ρ, η) -semispray associated to mechanical (ρ, η) -system $((E, \pi, M), F_e, (\rho, \eta)\Gamma)$ and from locally invertible \mathbf{B}^V -morphism (g, h) and let

$$(7.3)' \quad \dot{S} = p_b \left(g^{ab} \circ h \circ \pi^*\right) \dot{\tilde{\partial}}_a - 2 \left(G_a - \frac{1}{4} F_a\right) \dot{\tilde{\partial}}^a$$

be the (ρ, η) -semispray associated to the mechanical (ρ, η) -system $\left(\left(\begin{smallmatrix} * \\ E, \pi, M \end{smallmatrix}\right), F_e, (\rho, \eta)\Gamma\right)$ and from locally invertible \mathbf{B}^V -morphism (g, h) .

Theorem 7.1 *If $\Gamma((\rho, \eta)T\varphi_L, \varphi_L)(S) = \dot{S}$, then we obtain:*

$$(7.4) \quad y^b (g_b^a \circ h \circ \pi) \circ \varphi_H = p_b \left(g^{ab} \circ h \circ \pi^*\right)$$

and

$$(7.5) \quad \begin{aligned} 2(G_b - \tfrac{1}{4}F_b) &= 2[(G^a - \tfrac{1}{4}F^a) L_{ab}] \circ \varphi_H \\ &\quad - y^c \{[(g_c^a \rho_a^i) \circ h \circ \pi] L_{ib}\} \circ \varphi_H. \end{aligned}$$

Corollary 7.1 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$(7.4)' \quad \begin{aligned} 2(G_b - \tfrac{1}{4}F_b) &= 2[(G^a - \tfrac{1}{4}F^a) L_{ab}] \circ \varphi_H \\ &\quad - y^c \{[(g_c^a \rho_a^i) \circ \pi] L_{ib}\} \circ \varphi_H. \end{aligned}$$

In the classical case, $(\rho, \eta, h) = (Id_{TM}, Id_M, Id_M)$, we obtain the equality implies the equality

$$(7.5)'' \quad \begin{aligned} 2(G_i - \tfrac{1}{4}F_i) &= 2[(G^i - \tfrac{1}{4}F^i) \frac{\partial^2 L}{\partial y^i \partial y^j}] \circ \varphi_H \\ &\quad - y^j \frac{\partial^2 L}{\partial x^i \partial y^j} \circ \varphi_H. \end{aligned}$$

Theorem 7.2 *Dual, if $\Gamma((\rho, \eta) T\varphi_H, \varphi_H) \binom{*}{S} = S$, then we obtain:*

$$(7.6) \quad p_b (g^{ba} \circ h \circ \pi^*) \circ \varphi_L = y^b (g_b^a \circ h \circ \pi)$$

and

$$(7.7) \quad \begin{aligned} 2(G^a - \tfrac{1}{4}F^a) &= 2[(G_b - \tfrac{1}{4}F_b) H^{ab}] \circ \varphi_L \\ &\quad - p_c \{[(g^{ac} \rho_a^i) \circ h \circ \pi^*] H_i^b\} \circ \varphi_L. \end{aligned}$$

Corollary 7.2 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$(7.7)' \quad \begin{aligned} 2(G^a - \tfrac{1}{4}F^a) &= 2[(G_b - \tfrac{1}{4}F_b) H^{ab}] \circ \varphi_L \\ &\quad - p_c \{[(g^{ac} \rho_a^i) \circ \pi^*] H_i^b\} \circ \varphi_L. \end{aligned}$$

8 Duality between Lagrange and Hamilton mechanical (ρ, η) -systems

Let $((E, \pi, M), F_e, L)$ be an arbitrarily Lagrange mechanical (ρ, η) -system.

Let $d^{(\rho, \eta)TE}$ be the exterior differentiation operator associated to the exterior differential $\mathcal{F}(E)$ -algebra

$$(\Lambda((\rho, \eta) TE, (\rho, \eta) \tau_E, E), +, \cdot, \wedge)$$

and let (g, h) be a locally invertible \mathbf{B}^v -morphism of (E, π, M) source and (E, π, M) target.

Let $\left(\binom{*}{E, \pi, M}, F_e^*, H\right)$ be an Hamilton mechanical (ρ, η) -system, where the regular Hamiltonian H is the Legendre transformation of the regular Lagrangian L .

Let $d^{(\rho, \eta)TE^*}$ be the exterior differentiation operator associated to the exterior differential $\mathcal{F}\left(\binom{*}{E}\right)$ -algebra

$$\left(\Lambda\left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, \binom{*}{E}\right), +, \cdot, \wedge\right).$$

and let (g, h) be a locally invertible $\mathbf{B}^{\mathbf{v}}$ -morphism of $\left(E, \pi, M\right)^{\ast}$ source and (E, π, M) target.

The 1-form

$$(8.1) \quad \theta_L = (\tilde{g}_a^e \circ h \circ \pi \cdot L_e) d\tilde{z}^a$$

is called the 1-form of Poincaré-Cartan type associated to the Lagrangian L and to the locally invertible $\mathbf{B}^{\mathbf{v}}$ -morphism (g, h) . We obtain easily:

$$(8.2) \quad \theta_L \left(\tilde{\partial}_a \right) = (\tilde{g}_b^e \circ h \circ \pi) L_e, \quad \theta_L \left(\dot{\tilde{\partial}}_b \right) = 0.$$

The 1-form

$$(8.1)' \quad \theta_H = \left(\tilde{g}_{ae} \circ h \circ \pi^{\ast} \right) H^e d\tilde{z}^a$$

will be called the 1-form of Poincaré-Cartan type associated to the regular Hamiltonian H and from locally invertible $\mathbf{B}^{\mathbf{v}}$ -morphism (g, h) . We obtain easily:

$$(8.2)' \quad \theta_H \left(\tilde{\partial}_b^{\ast} \right) = \left(\tilde{g}_{be} \circ h \circ \pi^{\ast} \right) H^e, \quad \theta_H \left(\dot{\tilde{\partial}}^b \right) = 0.$$

Theorem 8.1 If $((\rho, \eta) T\varphi_L, \varphi_L)^{\ast}(\theta_L) = \theta_H$, then

$$(8.3) \quad \left(\tilde{g}_{ae} \circ h \circ \pi^{\ast} \right) H^e \circ \varphi_L = (\tilde{g}_b^e \circ h \circ \pi) L_e$$

Dual, if $((\rho, \eta) T\varphi_H, \varphi_H)^{\ast}(\theta_H) = \theta_L$, then

$$(8.3)' \quad (\tilde{g}_b^e \circ h \circ \pi) L_e \circ \varphi_H = \left(\tilde{g}_{ae} \circ h \circ \pi^{\ast} \right) H^e.$$

The 2-form

$$(8.4) \quad \omega_L = d^{(\rho, \eta)TE} \theta_L$$

is called the 2-form of Poincaré-Cartan type associated to the Lagrangian L and to the locally invertible $\mathbf{B}^{\mathbf{v}}$ -morphism (g, h) . By the definition of $d^{(\rho, \eta)TE}$, we obtain:

$$(8.5) \quad \begin{aligned} \omega_L(U, V) &= \Gamma(\tilde{\rho}, Id_E)(U)(\theta_L(V)) \\ &\quad - \Gamma(\tilde{\rho}, Id_E)(V)(\theta_L(U)) - \theta_L([U, V]_{(\rho, \eta)TE}), \end{aligned}$$

for any $U, V \in \Gamma((\rho, \eta)TE, (\rho, \eta)\tau_E, E)$.

The 2-form

$$(8.4)' \quad \omega_H = d^{(\rho, \eta)TE^{\ast}} \theta_H$$

will be called the 2-form of Poincaré-Cartan type associated to the Hamiltonian H and to the locally invertible $\mathbf{B}^{\mathbf{v}}$ -morphism (g, h) . By the definition of $d^{(\rho, \eta)TE^{\ast}}$, we obtain:

$$\begin{aligned}
(8.5)' \quad \omega_H(U, V) &= \Gamma \left(\tilde{\rho}, Id_E^* \right) (U) (\theta_H(V)) \\
&\quad - \Gamma \left(\tilde{\rho}, Id_E^* \right) (V) (\theta_H(U)) - \theta_H \left([U, V]_{(\rho, \eta) TE^*} \right),
\end{aligned}$$

for any $U, V \in \Gamma \left((\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right)$.

Theorem 8.2 *If $((\rho, \eta) T\varphi_L, \varphi_L)^* (\omega_L) = \omega_H$, then*

$$\begin{aligned}
(8.6) \quad & \left[\left(\rho_a^i \circ h \circ \pi^* \right) \frac{\partial((\tilde{g}_{be} \circ h \circ \pi^*) H^e)}{\partial x^i} \right] \circ \varphi_L - \left[\left(\rho_b^j \circ h \circ \pi^* \right) \frac{\partial((\tilde{g}_{ae} \circ h \circ \pi^*) H^e)}{\partial x^j} \right] \circ \varphi_L \\
& \left[\left(L_{ab}^c \circ h \circ \pi^* \right) (\tilde{g}_{ce} \circ h \circ \pi^*) H^e \right] \circ \varphi_L - \left[\left(\left(\rho_b^j \circ h \circ \pi^* \right) L_{jd} \circ \varphi_H \right) \frac{\partial((\tilde{g}_{be} \circ h \circ \pi^*) H^e)}{\partial p_d} \right] \circ \varphi_L \\
& - \left[\left(\left(\rho_a^i \circ h \circ \pi^* \right) L_{id} \right) \circ \varphi_H \frac{\partial((\tilde{g}_{ae} \circ h \circ \pi^*) H^e)}{\partial p_d} \right] \circ \varphi_L \\
& = \left(\rho_a^i \circ h \circ \pi \right) \frac{\partial((\tilde{g}_b^e \circ h \circ \pi) L_e)}{\partial x^i} - \left(\rho_b^j \circ h \circ \pi \right) \frac{\partial((\tilde{g}_a^e \circ h \circ \pi) L_e)}{\partial x^j} - (L_{ab}^c \circ h \circ \pi) (\tilde{g}_c^e \circ h \circ \pi) L_e
\end{aligned}$$

and

$$(8.7) \quad \left[L_{bc} \circ \varphi_H \frac{\partial((\tilde{g}_{ae} \circ h \circ \pi^*) H^e)}{\partial p_c} \right] \circ \varphi_L = \frac{\partial((\tilde{g}_a^e \circ h \circ \pi) L_e)}{\partial y^b}.$$

Corollary 8.1 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$\begin{aligned}
(8.6)' \quad & \left[\left(\rho_a^i \circ \pi^* \right) \frac{\partial((\tilde{g}_{be} \circ \pi^*) H^e)}{\partial x^i} \right] \circ \varphi_L - \left[\left(\rho_b^j \circ \pi^* \right) \frac{\partial((\tilde{g}_{ae} \circ \pi^*) H^e)}{\partial x^j} \right] \circ \varphi_L \\
& \left[\left(L_{ab}^c \circ \pi^* \right) (\tilde{g}_{ce} \circ \pi^*) H^e \right] \circ \varphi_L - \left[\left(\left(\rho_b^j \circ \pi^* \right) L_{jd} \circ \varphi_H \right) \frac{\partial((\tilde{g}_{be} \circ \pi^*) H^e)}{\partial p_d} \right] \circ \varphi_L \\
& - \left[\left(\left(\rho_a^i \circ \pi^* \right) L_{id} \right) \circ \varphi_H \frac{\partial((\tilde{g}_{ae} \circ \pi^*) H^e)}{\partial p_d} \right] \circ \varphi_L \\
& = \left(\rho_a^i \circ \pi \right) \frac{\partial((\tilde{g}_b^e \circ \pi) L_e)}{\partial x^i} - \left(\rho_b^j \circ \pi \right) \frac{\partial((\tilde{g}_a^e \circ \pi) L_e)}{\partial x^j} - (L_{ab}^c \circ \pi) (\tilde{g}_c^e \circ \pi) L_e
\end{aligned}$$

and

$$(8.7)' \quad \left[L_{bc} \circ \varphi_H \frac{\partial((\tilde{g}_{ae} \circ \pi^*) H^e)}{\partial p_c} \right] \circ \varphi_L = \frac{\partial((\tilde{g}_a^e \circ \pi) L_e)}{\partial y^b}.$$

Theorem 8.3 *If $((\rho, \eta) T\varphi_H, \varphi_H)^* (\omega_H) = \omega_L$, then*

$$\begin{aligned}
(8.8) \quad & \left[\left(\rho_a^i \circ h \circ \pi \right) \frac{\partial((\tilde{g}_b^e \circ h \circ \pi) L_e)}{\partial x^i} \right] \circ \varphi_H - \left[\left(\rho_b^j \circ h \circ \pi \right) \frac{\partial((\tilde{g}_a^e \circ h \circ \pi) L_e)}{\partial x^j} \right] \circ \varphi_H \\
& \left[(L_{ab}^c \circ h \circ \pi) (\tilde{g}_c^e \circ h \circ \pi) L_e \right] \circ \varphi_H - \left[\left(\left(\rho_b^j \circ h \circ \pi \right) H_j^d \circ \varphi_L \right) \frac{\partial((\tilde{g}_b^e \circ h \circ \pi) L_e)}{\partial y^d} \right] \circ \varphi_H \\
& - \left[\left(\left(\rho_a^i \circ h \circ \pi \right) H_i^d \circ \varphi_L \right) \frac{\partial((\tilde{g}_a^e \circ h \circ \pi) L_e)}{\partial y^d} \right] \circ \varphi_H \\
& = \left(\rho_a^i \circ h \circ \pi \right) \frac{\partial((\tilde{g}_{be} \circ h \circ \pi^*) H^e)}{\partial x^i} - \left(\rho_b^j \circ h \circ \pi \right) \frac{\partial((\tilde{g}_{ae} \circ h \circ \pi^*) H^e)}{\partial x^j} - (L_{ab}^c \circ h \circ \pi) (\tilde{g}_{ce} \circ h \circ \pi^*) H^e
\end{aligned}$$

and

$$(8.9) \quad \left[H^{bc} \circ \varphi_L \frac{\partial((\tilde{g}_a^e \circ h \circ \pi) L_e)}{\partial y^c} \right] \circ \varphi_H = \frac{\partial((\tilde{g}_{ae} \circ h \circ \pi^*) H^e)}{\partial p_b}.$$

Corollary 8.2 *In the particular case of Lie algebroids, $(\eta, h) = (Id_M, Id_M)$, we obtain*

$$(8.8)' \quad \begin{aligned} & \left[(\rho_a^i \circ \pi) \frac{\partial((\tilde{g}_b^e \circ \pi) L_e)}{\partial x^i} \right] \circ \varphi_H - \left[(\rho_b^j \circ \pi) \frac{\partial((\tilde{g}_a^e \circ \pi) L_e)}{\partial x^j} \right] \circ \varphi_H \\ & [(L_{ab}^c \circ \pi) (\tilde{g}_c^e \circ \pi) L_e] \circ \varphi_H - \left[\left((\rho_b^j \circ \pi) H_j^d \circ \varphi_L \right) \frac{\partial((\tilde{g}_b^e \circ \pi) L_e)}{\partial y^d} \right] \circ \varphi_H \\ & - \left[\left((\rho_a^i \circ \pi) H_i^d \circ \varphi_L \right) \frac{\partial((\tilde{g}_a^e \circ \pi) L_e)}{\partial y^d} \right] \circ \varphi_H \\ & = \left(\rho_a^i \circ \pi^* \right) \frac{\partial((\tilde{g}_{be} \circ \pi^*) H^e)}{\partial x^i} - \left(\rho_b^j \circ \pi^* \right) \frac{\partial((\tilde{g}_{ae} \circ \pi^*) H^e)}{\partial x^j} - \left(L_{ab}^c \circ \pi^* \right) (\tilde{g}_{ce} \circ \pi^*) H^e \end{aligned}$$

and

$$(8.9)' \quad \left[H^{bc} \circ \varphi_L \frac{\partial((\tilde{g}_a^e \circ \pi) L_e)}{\partial y^c} \right] \circ \varphi_H = \frac{\partial((\tilde{g}_{ae} \circ \pi^*) H^e)}{\partial p_b}.$$

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